Phys 410
Fall 2014
Lecture #6 Summary
18 September, 2014

We did an example of a Hooke's law restoring force in 1D: $\vec{F} = -k\,x\,\hat{x}$, with an equilibrium point at $x = 0$. The corresponding potential is $U(x) = \frac{1}{2}kx^2$, with $U(0) = 0$. The mechanical energy is conserved: $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$. As the particle moves it exchanges energy back and forth between kinetic and potential energies. Note that the force can be derived from the potential energy function through the 1D gradient, which is a total derivative: $F_{\tan} = -dU(x)/dx$.

The energy landscape created by the function $U(x)$ is very revealing. If there is a maximum or minimum in $U(x)$ it means that the driving force at that location is $F_{\tan} = 0$. As such, this represents an equilibrium point. A minimum in $U(x)$ is a stable equilibrium because a small displacement will result in forces that point back to the equilibrium point. This is the case when $d^2U(x)/dx^2 > 0$. A local maximum in $U(x)$ is unstable because a small displacement in either direction produces forces that draw the particle further away. This is the case with $d^2U(x)/dx^2 < 0$.

One-dimensional problems, although they appear to be artificial, pop up frequently in the solution of three-dimensional problems. So far we have no mention of time in the evolution. We can find the position of the particle $x(t)$ starting with the mechanical energy, and at least one additional piece of information (the sign of $\dot{x}$). We showed that the statement of mechanical energy conservation can be re-written as $t = \sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{E - U(x')}}$. This can be solved for $x(t)$, and from that one can determine the velocity, acceleration, etc. as functions of time. We did problem 4.28 from HW 3 and solved the above equation for $x(t)$ for a simple harmonic oscillator.

We considered energy for motion in curvilinear one-dimensional systems. An example is a car moving on a roller coaster track. Consider a particle confined to move along a one-dimensional 'track' parameterized by its displacement s from some arbitrary origin. It has a kinetic energy $T = \frac{1}{2}m\dot{s}^2$. The kinetic energy can be altered by applying a tangential force and doing work on the particle. Newton's second law can be stated as $m\ddot{s} = F_{net}$. If the tangential force is conservative, then you can define a potential energy U, and a total mechanical energy $E = T + U$.

We next considered central forces. These are forces that are everywhere directed toward a fixed force center. Such a force has the form $\vec{F}(\vec{r}) = f(\vec{r})\hat{r}$. If further the force is spherically symmetric, then the scalar function depends only on the radial distance and not the angular coordinates: $f(\vec{r}) = f(r)$.

There are two statements that can be made about central forces:

1) A central force that is conservative is automatically spherically symmetric,
2) A central force that is spherically symmetric is automatically conservative.

We proved the first of these two statements. If the force is conservative, then it can be represented in terms of the gradient of a scalar potential: $\vec{F} = -\vec{\nabla}U(\vec{r})$. Using the gradient in spherical coordinates, derived in class ($\vec{\nabla} = \hat{r}\partial/\partial r + (\hat{\theta}/r)\partial/\partial\theta + (\hat{\phi}/r\sin\theta)\partial/\partial\phi$), we find that a central force (dependent on $\hat{r}$ only) requires that $\partial U/\partial\theta = \partial U/\partial\phi = 0$. This means that the potential energy depends only on the radial coordinate: $U = U(r)$. In turn, the central force can only depend on the scalar radial coordinate: $\vec{F} = -\hat{r}\ \partial U(r)/\partial r$, which means that it is spherically symmetric (i.e. no dependence of the potential and force on the angular coordinates θ, ϕ). The one-dimensional nature of the potential energy and force will have benefits later when we look at the two-body problem.